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DYNAMIC FINITE ELEMENT MODEL UPDATING USING AMBIENT VIBRATION MEASUREMENTS

Hua-Peng Chen
School of Engineering, University of Greenwich, Chatham Maritime, Kent, ME4 4TB, UK.
Email: h.chen@gre.ac.uk

ABSTRACT

This paper presents an effective approach for updating finite element dynamic model from measured incomplete modal data identified from ambient vibration measurements. The proposed method is based on the relationship between the perturbation of structural parameters such as stiffness change and the modal data measurements of the tested structure such as measured mode shape readings. Structural updating parameters characterising stiffness and mass matrices at element level are employed to represent the differences in structural parameters between the finite element model and the associated tested structure. These updating parameters are then evaluated by an iterative solution procedure, which gives optimised solutions in the least squares sense without requiring an optimisation technique. In order to reduce the influence of modal measurement uncertainty, the Tikhonov regularization method incorporating the L-curve criterion is employed to produce reliable solutions for the structural updating parameters. Finally, the practical applications to the Canton Tower benchmark problem demonstrate that the proposed method can correctly update the given finite element model using the incomplete modal data identified from the recorded ambient vibration measurements by an operational modal analysis technique.

KEYWORDS

Finite element model updating, dynamic perturbation method, regularization algorithm, ambient vibration measurements, Canton Tower benchmark problem.

INTRODUCTION

The finite element (FE) method is useful for many applications in engineering practice such as structural analysis and structural damage detection (Brownjohn and Xia 2000; Chen 2006; Chen 2010). In general, significant discrepancies may exist between numerical predictions from the numerical model and experimental results obtained from the actual tested structure. Those discrepancies are mainly caused by the uncertainties in the numerical model in modelling physical aspects of the actual built structure, such as material properties, imperfection in geometry and boundary conditions.

Extensive FE model updating methods using vibration measurements have been undertaken during the last two decades (Mottershead and Friswell 1993). The model updating methods generally fall under two major categories, i.e. direct methods and iterative methods. The direct methods, also called model-based methods, are a one-step procedure and directly reconstruct the elements stiffness and mass matrices (Caesar and Peter 1987; Friswell *et al.* 1998). In these methods, incomplete measured modal data could be used for updating structural parameters without using standard modal expansion or model reduction techniques (Carvalho *et al*. 2007). However, the applicability of the direct methods may be restricted in the case of updating the large number of structural parameters and the updated physical properties may not truly represent the actual tested structure concerned. On the other hand, iterative updating methods including sensitivity-based approaches are widely used in FE model updating. In these methods, the errors between the initial FE model predictions and measured data are set as objective function and attempt to minimize the selected objective function by adjusting the pre-selected set of physical parameters of the numerical model (Modak *et al*. 2002; Chen 2005). However, the sensitivity analysis used in the iterative model updating methods may not perform well, in particular cases when large modification of structural parameters is required (Chen and Huang 2012).

This paper presents a reliable model updating method which can directly adopt measured incomplete modal data for evaluating the chosen structural updating parameters of the initial FE model, without requiring mode shape

expansion or model reduction techniques. On the basis of the dynamic perturbation method, the exact relationship between the structural parameter modifications and the incomplete measured modal data of the tested structure is generated, without requiring sensitivity analysis and the construction of an objective function. The structural parameters to be updated are chosen to characterise the modifications of structural parameters, i.e. differences in stiffness and mass matrices between the numerical model and tested structure at element level. An iterative solution procedure is proposed to estimate the chosen structural updating parameters in the least squares sense, and no optimisation technique is required. The Tikhonov regularization algorithm incorporating the L-curve criterion method for determining the regularization parameter is employed to reduce the influence of modal measurement uncertainty and then to produce reliable solutions for the structural updating parameters. The effectiveness and applicability of the proposed FE model updating method are demonstrated by the applications to the updating of the given FE model of the actually constructed structure associated with the Canton Tower benchmark problem.

GOVERNING EQUATIONS FOR MODEL UPDATING

Dynamic Perturbation Method

For an undamped dynamic structural system with global stiffness matrix **K** and mass matrix **M**, the characteristic equation of an *N* Degrees of Freedom (DOFs) FE model is given by

$$(\mathbf{K} - \omega_i^{\,2}\mathbf{M})\boldsymbol{\phi}_i = \mathbf{0} \tag{1}$$

where ω_i and $\boldsymbol{\phi}_i$ are the *i*th frequency and the corresponding eigenvector of the FE model, respectively. The numerical eigenvectors are assumed to be normalised as unity with respect to the mass of the FE model. The FE model often has uncertainty in modelling structural parameters such as the stiffness and mass of the associated tested structure due to the complexity of the actually constructed structure. The model uncertainty is mainly related to the unknown perturbations of structural parameters such as differences in stiffness ($\Delta\mathbf{K}$) and mass ($\Delta\mathbf{M}$) between the FE model and the tested structure. The global stiffness matrix ($\hat{\mathbf{K}}$) and mass matrix ($\hat{\mathbf{M}}$) of the tested dynamic structure is expressed as

$$\hat{\mathbf{K}} = \mathbf{K} + \Delta\mathbf{K} \tag{2a}$$

$$\hat{\mathbf{M}} = \mathbf{M} + \Delta\mathbf{M} \tag{2b}$$

The unknown perturbations of structural parameters $\Delta\mathbf{K}$ and $\Delta\mathbf{M}$ can be described as

$$\Delta\mathbf{K} = \sum_{j=1}^{N\alpha} \alpha_j \mathbf{K}_j \quad , \quad \Delta\mathbf{M} = \sum_{m=1}^{N\beta} \beta_m \mathbf{M}_m \tag{3}$$

where $N\alpha$ and $N\beta$ represents the total number of stiffness and mass updating parameters adopted in calculations; α_j and β_j are stiffness and mass updating parameters characterised at element level; $\mathbf{K}_j$ and $\mathbf{M}_j$ are the contribution of the j^{th} element to the global stiffness and mass matrices, respectively.

Similarly, the characteristic equation for the actual tested structure is given as

$$(\hat{\mathbf{K}} - \hat{\omega}_i^{\,2}\hat{\mathbf{M}})\hat{\boldsymbol{\phi}}_i = \mathbf{0} \tag{4}$$

in which $\hat{\omega}_i$ and $\hat{\boldsymbol{\phi}}_i$ are the *i*th measured natural frequency and the corresponding mode shape for the tested dynamic system, respectively. The complete mode shapes of the tested structure with dimension of *N* can be expressed as a linear combination of the numerical eigenvectors, since the numerical eigenvectors are linearly independent due to the symmetry of the stiffness and mass matrices of the FE model, namely

$$\hat{\boldsymbol{\phi}}_i = \sum_{k=1}^{N} C_{ik}\boldsymbol{\phi}_k \tag{5}$$

where C_{ik} are mode participation factors. Premultiplying Eq. (5) by $\boldsymbol{\phi}_k^{\,T}\mathbf{M}$ and using the mass normalisation of the numerical eigenvectors, yields

$$C_{ik} = \boldsymbol{\phi}_k^{\,T}\mathbf{M}\hat{\boldsymbol{\phi}}_i \tag{6}$$

Assuming that the mode shapes of the tested structure are scaled in the following form

$$\boldsymbol{\phi}_i^{\,T}\mathbf{M}\hat{\boldsymbol{\phi}}_i = 1 \tag{7}$$

the mode participation factor C_{ii} becomes $C_{ii} = 1$. The mode shapes of the tested structure in Eq. (5) is then rewritten as

$$\hat{\boldsymbol{\phi}}_i = \boldsymbol{\phi}_i + \sum_{k=1,k\neq i}^{N} C_{ik}\boldsymbol{\phi}_k \tag{8}$$

From Eq. (2), the experimental characteristic equation in Eq. (3) becomes

$$[(\Delta\mathbf{K} - \hat{\omega}_i^{\,2}\Delta\mathbf{M}) + (\mathbf{K} - \hat{\omega}_i^{\,2}\mathbf{M})]\hat{\boldsymbol{\phi}}_i = \mathbf{0} \tag{9}$$

Pre-multiplying Eq. (9) by $\boldsymbol{\phi}_k^{\,T}$ and using Eqs. (6) and (8), the mode participation factors C_{ik} are now calculated from

$$C_{ik} = \frac{\boldsymbol{\phi}_k^{\,T}(\Delta\mathbf{K} - \hat{\omega}_i^{\,2}\Delta\mathbf{M})\hat{\boldsymbol{\phi}}_i}{\hat{\omega}_i^{\,2} - \omega_k^{\,2}} \tag{10}$$

Substituting the mode participation factors C_{ik} in Eq. (10) into Eq. (5), yields

$$\sum_{k=1}^{N} \frac{\boldsymbol{\phi}_k^{\,T}(\Delta\mathbf{K} - \hat{\omega}_i^{\,2}\Delta\mathbf{M})\hat{\boldsymbol{\phi}}_i}{\hat{\omega}_i^{\,2} - \omega_k^{\,2}}\boldsymbol{\phi}_k - \hat{\boldsymbol{\phi}}_i = \mathbf{0} \tag{11}$$

In structural dynamic testing, modal data about the natural frequency $\hat{\omega}_i$ and mode shape readings $\hat{\boldsymbol{\psi}}_i$ of the tested structure can be extracted from field vibration measurements by operational modal analysis techniques. The measured mode shapes $\hat{\boldsymbol{\psi}}_i$, with dimension of *Ns* where *Ns* is the total number of effective sensors installed, are usually incomplete with reference to the FE model typically having large number of DOFs, since the number of sensors installed for vibration measurements is often limited in practice. The measured modes are paired to the numerical eigenvectors of dimension *Ns* (restricted to the same dimensions as $\hat{\boldsymbol{\psi}}_i$), $\boldsymbol{\phi}_k^a$, by using Modal Assurance Criterion (MAC) factors defined as

$$MAC(\boldsymbol{\phi}_k^a, \hat{\boldsymbol{\psi}}_i) = \frac{\left|\boldsymbol{\phi}_k^{a\,T}\hat{\boldsymbol{\psi}}_i\right|^2}{\left|\boldsymbol{\phi}_k^{a\,T}\boldsymbol{\phi}_k^a\right|\left|\hat{\boldsymbol{\psi}}_i^{\,T}\hat{\boldsymbol{\psi}}_i\right|} \tag{12}$$

From above definition, large MAC factors indicate a high degree of similarity between two mode shapes, while small MAC factors represent little or even no correlation between two vectors. In the application of governing equation Eq. (11), the measured incomplete mode shapes of the actual tested structure need to be close to the corresponding part of the original numerical eigenvectors $\boldsymbol{\phi}_i^a$. The measured DOF's readings $\hat{\boldsymbol{\psi}}_i$ are then scaled by a mode scale factor υ_i, giving the scaled modal measurement vector $\hat{\boldsymbol{\phi}}_i^a$ of dimension *Ns* as

$$\hat{\boldsymbol{\phi}}_i^a = \upsilon_i\hat{\boldsymbol{\psi}}_i \;,\quad where\;\; \upsilon_i = \frac{\left|\boldsymbol{\phi}_i^{a\,T}\hat{\boldsymbol{\psi}}_i\right|}{\left|\hat{\boldsymbol{\psi}}_i^{\,T}\hat{\boldsymbol{\psi}}_i\right|} \tag{13}$$

The remaining (*N-Ns*) unmeasured components $\hat{\boldsymbol{\phi}}_i^u$, by using Eq. (8), are calculated from

$$\hat{\boldsymbol{\phi}}_i^u = \boldsymbol{\phi}_i^u + \sum_{k=1,k\neq i}^{N} C_{ik}\boldsymbol{\phi}_k^u \tag{14}$$

where $\boldsymbol{\phi}_k^u$ is original eigenvector corresponding to the entries of the unmeasured components. For simplicity and convenience in matrix operations, both the measured components $\hat{\boldsymbol{\phi}}_i^a$ and the remaining calculated components $\hat{\boldsymbol{\phi}}_i^u$ of the *i*th mode shape of tested structure are expanded to full dimension with zeroes at their unknown entries. The *i*th complete mode shape of the tested structure, consisting of the measured components and the remaining calculated components, is then given by

$$\hat{\boldsymbol{\phi}}_i = \boldsymbol{\varphi}_i^a + \sum_{k=1,k\neq i}^{N} C_{ik}\boldsymbol{\phi}_k^u \tag{15}$$

where $\boldsymbol{\varphi}_i^a$ is a known vector of dimension *N*, defined as

$$\boldsymbol{\varphi}_i^a = \upsilon_i \hat{\boldsymbol{\psi}}_i + \boldsymbol{\phi}_i^u \tag{16}$$

From Eq. (15), the mode participation factors C_{ik} in Eq. (10) are rewritten as

$$C_{ik} = \frac{\boldsymbol{\phi}_k{}^T \Delta\mathbf{K}(\boldsymbol{\varphi}_i^a + \sum_{l=1,l\neq i}^{N} C_{il}\boldsymbol{\phi}_l^u) - \hat{\omega}_i{}^2 \boldsymbol{\phi}_k{}^T \Delta\mathbf{M}(\boldsymbol{\varphi}_i^a + \sum_{l=1,l\neq i}^{N} C_{il}\boldsymbol{\phi}_l^u)}{\hat{\omega}_i{}^2 - \omega_k{}^2} \tag{17}$$

By using the constructed eigenvector in Eq. (15), Equation (11) is now restricted to the dimension for the measured components and becomes

$$\sum_{k=1}^{N} \frac{\boldsymbol{\phi}_k{}^T \Delta\mathbf{K}(\boldsymbol{\varphi}_i^a + \sum_{l=1,l\neq i}^{N} C_{il}\boldsymbol{\phi}_l^u)}{\hat{\omega}_i{}^2 - \omega_k{}^2}\boldsymbol{\phi}_k^a - \hat{\omega}_i{}^2 \sum_{k=1}^{N} \frac{\boldsymbol{\phi}_k{}^T \Delta\mathbf{M}(\boldsymbol{\varphi}_i^a + \sum_{l=1,l\neq i}^{N} C_{il}\boldsymbol{\phi}_l^u)}{\hat{\omega}_i{}^2 - \omega_k{}^2}\boldsymbol{\phi}_k^a - \hat{\boldsymbol{\phi}}_i^a = \mathbf{0} \tag{18}$$

It should be noted that the governing equations in Eq. (18) represents the exact relationship between the change in structural parameters and the measured modal properties of the tested structure such as mode shape readings. The governing equations avoid the approximations and complexity in most existing eigensesitivity-based analysis methods for model updating, without requiring eigensolution derivatives.

Solution Procedure for Updating Parameters

In order to minimise the computational effort, by using the unknown perturbation of structural parameters $\Delta\mathbf{K}$ and $\Delta\mathbf{M}$ in Eq. (3), the sensitivity coefficients associated with the known eigenmodes and structural parameters in the governing equation Eq. (18) are defined in a general form as

$$\mathbf{p}_{jii}^a = \sum_{k=1}^{N} \frac{\boldsymbol{\phi}_k{}^T \mathbf{K}_j \boldsymbol{\varphi}_i^a}{\hat{\omega}_i{}^2 - \omega_k{}^2}\boldsymbol{\phi}_k^a \quad , \quad \mathbf{p}_{jil}^u = \sum_{k=1}^{N} \frac{\boldsymbol{\phi}_k{}^T \mathbf{K}_j \boldsymbol{\phi}_l^u}{\hat{\omega}_i{}^2 - \omega_k{}^2}\boldsymbol{\phi}_k^a \tag{19a}$$

$$\mathbf{q}_{mii}^a = \sum_{k=1}^{N} \frac{\boldsymbol{\phi}_k{}^T \mathbf{M}_m \boldsymbol{\varphi}_i^a}{\hat{\omega}_i{}^2 - \omega_k{}^2}\boldsymbol{\phi}_k^a \quad , \quad \mathbf{q}_{mil}^u = \sum_{k=1}^{N} \frac{\boldsymbol{\phi}_k{}^T \mathbf{M}_m \boldsymbol{\phi}_l^u}{\hat{\omega}_i{}^2 - \omega_k{}^2}\boldsymbol{\phi}_k^a \tag{19b}$$

therefore, the governing equation Eq. (18) for the *i*th measured mode is rewritten as

$$\sum_{j=1}^{N\alpha} [\mathbf{p}_{jii}^a + \sum_{k=1,k\neq i}^{N} \mathbf{p}_{jik}^u C_{ik}]\alpha_j - \hat{\omega}_i{}^2 \sum_{m=1}^{N\beta} [\mathbf{q}_{mii}^a + \sum_{k=1,k\neq i}^{N} \mathbf{q}_{mik}^u C_{ik}]\beta_m - \hat{\boldsymbol{\phi}}_i^a = \mathbf{0} \tag{20}$$

Similarly, define the sensitivity coefficients associated with the known eigenmodes and structural parameters in Eq. (17) in a general form as

$$g_{kji}^a = \boldsymbol{\phi}_k^T \mathbf{K}_j \boldsymbol{\varphi}_i^a \quad , \quad g_{kjl}^u = \boldsymbol{\phi}_k^T \mathbf{K}_j \boldsymbol{\phi}_l^u \tag{21a}$$

$$h_{kmi}^a = \boldsymbol{\phi}_k^T \mathbf{M}_m \boldsymbol{\varphi}_i^a \quad , \quad h_{kml}^u = \boldsymbol{\phi}_k^T \mathbf{M}_m \boldsymbol{\phi}_l^u \tag{21b}$$

the mode participation factors C_{ik} for the *i*th measured incomplete mode shape in Eq. (17) are expressed as

$$C_{ik} = \frac{\sum_{j=1}^{N\alpha} [g_{kji}^a + \sum_{l=1,l\neq i}^{N} g_{kjl}^u C_{il}]\alpha_j - \hat{\omega}_i{}^2 \sum_{m=1}^{N\beta} [h_{kmi}^a + \sum_{l=1,l\neq i}^{N} h_{kml}^u C_{il}]\beta_m}{\hat{\omega}_i{}^2 - \omega_k{}^2} \tag{22}$$

On the basis of the governing equations Eq. (20) and Eq. (22) developed above, an iterative solution procedure is required to solve for a total number of $Nx\,(=N\alpha+N\beta)$ structural updating parameters α_j and β_m. In the case when a total number of *Nm* experimental modes are measured from a total number of *Ns* sensors installed on the tested structure, the governing equation Eq. (20) should be used for the total number of *Nm* measured modes to generate a total number of $Np\,(=Nm\times Ns)$ equations available in order to find a solution for a total number of *Nx* structural updating parameters to be determined.

The iterative solution procedure is initiated by assuming that the initial mode participation factors are zero, i.e. $C_{ik}{}^{(0)} = 0,\ (k \neq i)$. The first approximation for the structural updating parameters $\alpha_j{}^{(1)}$ and $\beta_m{}^{(1)}$ then can be obtained from the constructed governing equations, which now become a set of linear equations because of

the known $C_{ik}^{(0)}$. A regularization method discussed in the following section is applied to find a solution for what is in general an ill-conditioned system. After the first estimate of structural updating parameters $\alpha_j^{(1)}$ and $\beta_m^{(1)}$ is obtained, the next approximation for $C_{ik}^{(1)}$ can be calculated from Eq. (22). Consequently, the set of governing equations is used recursively to compute the *n*th ($n \geq 2$) approximations for $\alpha_j^{(n)}$ and $\beta_m^{(n)}$ as well as $C_{ik}^{(n)}$. The above recursive process is repeated until the convergence for structural updating parameters $\alpha_j^{(n)}$ and $\beta_m^{(n)}$ is achieved, often after only a few iterations.

REGULARISED SOLUTION FOR MODEL UPDATING

A regularization algorithm is often required to obtain stable solutions for the ill-posed problem defined in the governing equations in Eq. (20). For the *i*th measured mode with a total of *Ns* mode shape readings, these governing equations are rewritten here as

$$\mathbf{A}^{(i)}\mathbf{x} = \mathbf{b}^{(i)}, \quad \mathbf{A}^{(i)} \in R^{Ns \times Nx}, \quad \mathbf{x} \in R^{Nx}, \quad \mathbf{b}^{(i)} \in R^{Ns} \tag{23}$$

where $\mathbf{A}^{(i)}$ is sensitivity matrix of dimensions *Ns*×*Nx* associated with sensitivity coefficients and the obtained mode participation factors defined in the governing equations Eq. (20) for the *i*th measured mode; **x** is unknown vector of dimension *Nx* containing structural updating parameters α_j and β_m; $\mathbf{b}^{(i)}$ is known vector of dimension *Ns* containing DOF's readings for the *i*th measured mode. In order to reduce the influence of measurement uncertainty on FE model updating, the Tikhonov regularization, one of the most popular regularization methods (Tikhonov and Arsenin 1977), is adopted in this study. In the case with a total number of *Nm* measured modes available, by using the linear equations in Eq. (23) for each measured mode, the Tikhonov regularization solution can be replaced by a minimisation problem

$$\min_{\mathbf{x} \in R^{Nx}} \{ \sum_{i=1}^{Nm} \left\| w_i^{\frac{1}{2}} (\mathbf{A}^{(i)}\mathbf{x} - \mathbf{b}^{(i)}) \right\|^2 + \lambda^2 \left\| \mathbf{x} \right\|^2 \} = \min_{\mathbf{x} \in R^{Nx}} \{ \left\| (\mathbf{A}\mathbf{x} - \mathbf{b}) \right\|^2 + \lambda^2 \left\| \mathbf{x} \right\|^2 \} \tag{24}$$

where w_i (*i*=1, *Nm*) is weighting factor reflecting the importance and/or accuracy of the *i*th measured modes; **A** is weighted sensitivity matrix of dimensions *Np*×*Nx* associated with the corresponding weighting factor, **b** is weighted known vector of dimension *Np* containing all measured modal shape readings; $\lambda \geq 0$ is the regularization parameter and $\left\| \bullet \right\|$ denotes the Euclidean norm. An explicit regularised solution $\mathbf{x}_\lambda$ to the minimisation problem is given by

$$\mathbf{x}_\lambda = (\mathbf{A}^T\mathbf{A} + \lambda^2\mathbf{I})^{-1}\mathbf{A}^T\mathbf{b} \tag{25}$$

where **I** is the identity matrix. Assume $Np \succ Nx$ (in the case when $Np \leq Nx$, the same procedure can be applied) and let the singular value decomposition (SVD) of the sensitivity coefficient matrix **A** be

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \sum_{j=1}^{Nx} \sigma_j \mathbf{u}_j \mathbf{v}_j^T \tag{26}$$

where $\mathbf{\Sigma}$ is a diagonal matrix $\mathbf{\Sigma} = diag[\sigma_1, \sigma_2, ..., \sigma_{Nx}]$ in which $\sigma_1 \geq \sigma_2 \geq, ..., \geq \sigma_{Nx}$; **U** and **V** are orthogonal matrices with $\mathbf{U}\mathbf{U}^T = \mathbf{U}^T\mathbf{U} = \mathbf{I}$ and $\mathbf{V}\mathbf{V}^T = \mathbf{V}^T\mathbf{V} = \mathbf{I}$. By using the SVD of **A** in Eq. (26), the regularised solution $\mathbf{x}_\lambda$ is thus given by

$$\mathbf{x}_\lambda = \mathbf{V}\mathbf{\Sigma}_\lambda^+\mathbf{U}^T\mathbf{b} = \sum_{j=1}^{Nx} \frac{\sigma_j}{\sigma_j^{\,2} + \lambda^2} \mathbf{u}_j^T\mathbf{b}\mathbf{v}_j \tag{27}$$

where $\mathbf{\Sigma}_\lambda^+$ is a diagonal matrix, defined as

$$\mathbf{\Sigma}_\lambda^+ = diag[\frac{\sigma_1}{\sigma_1^{\,2} + \lambda^2}, \frac{\sigma_2}{\sigma_2^{\,2} + \lambda^2}, ..., \frac{\sigma_{Nx}}{\sigma_{Nx}^{\,2} + \lambda^2}] \tag{28}$$

The regularization parameter λ in Eq. (27) needs to be properly selected in order to filter out enough noise without losing too much information in the regularised solution. Computational experience shows the L-curve criterion can provide suitable values of regularization parameters in many problems (Hansen and O'Leary 1993),

without requiring the *priori* knowledge of noise in the measured data. The L-curve is a plot in log-log scale of the corresponding values of the residual norm $\rho(\lambda)$ and solution norm $\eta(\lambda)$ as a function of the regularization parameter λ, defined in terms of the SVD as

$$\rho(\lambda) = \left\|\mathbf{A}\mathbf{x}_\lambda - \mathbf{b}\right\|^2 = \left\|(\mathbf{\Sigma}\mathbf{\Sigma}_\lambda^+ - \mathbf{I})\mathbf{u}_j{}^T\mathbf{b}\right\|^2 = \sum_{j=1}^{Nx}[\frac{\lambda^2}{\sigma_j{}^2 + \lambda^2}\mathbf{u}_j{}^T\mathbf{b}]^2 \tag{29a}$$

$$\eta(\lambda) = \left\|\mathbf{x}_\lambda\right\|^2 = \left\|\mathbf{\Sigma}_\lambda^+\mathbf{u}_j{}^T\mathbf{b}\right\|^2 = \sum_{j=1}^{Nx}[\frac{\sigma_j}{\sigma_j{}^2 + \lambda^2}\mathbf{u}_j{}^T\mathbf{b}]^2 \tag{29b}$$

It can be shown that the L-curve's corner, where the curvature of the L-curve approximately has a maximum value, indicates an optimal regularization parameter to balance the residual norm error and the solution norm error (Hansen and O'Leary 1993). From the definitions given in Eq. (29), the curvature of the L-curve $\kappa(\lambda)$ is obtained from

$$\kappa(\lambda) = \frac{2\eta\rho}{\eta'}\frac{(\lambda^2\eta'\rho + 2\lambda\eta\rho + \lambda^4\eta\eta')}{(\lambda^4\eta^2 + \rho^2)^{3/2}} \tag{30}$$

where η' denotes the first derivative of η with respect to λ. A one-dimensional optimisation procedure is adopted to determine the optimal regularization parameter λ corresponding to the maximum curvature. Therefore, the regularised solutions for structural updating parameters are obtained from Eq. (27) where λ is determined by maximising Eq. (30).

VERIFICATION BY PRACTICAL APPLICATIONS

The proposed model updating method is now applied to updating the numerical model of the actually constructed structure, the Canton Tower in Guangzhou, China. The tower shown in Figure 1(a) has a total height of 610m with a 454m high main tower and a 156m high antenna mast. A total number of 20 uni-axial accelerometers were installed at eight different levels, as shown in Figure 1(b). The detailed information on the structure and the implemented structural health monitoring system can be found in the study by Chen *et al*. (2011) and Ni *et al*. (2012). In order to carry out structural health monitoring and associated studies, a reduced-order 3D beam model with 38 nodes and 37 beam elements was developed on the basis of the complex 3D full FE model, as shown in Figure 1(c). The reduced numerical model was tuned so that its dynamic characteristics match those of the full FE model as closely as possible. The ambient vibration measurements were recorded through the installed system for 24 hours from 18:00pm on 19 January to 18:00pm on 20 January 2010 with a sampling frequency of 50 Hz. These measurements were released for the Phase I of the Benchmark Problem involving the updating of the reduced numerical model. The extracted acceleration measurements recorded by accelerometers 01 and 02 during the first 60 seconds after 21:00pm of the day are plotted in Figs. 1(d) and (e), respectively.

In this study, the ambient vibration measurements recorded from 21:00pm to 22:00pm are adopted to identify operational modal properties of the tower, such as frequencies and mode shape measurements, by using the stochastic subspace identification technique (Peeters and De Roeck 1999). The identified natural frequencies of the tower are indicated in the stabilisation diagram as shown in Figure 2. The modal properties identified from ambient vibration measurements, together with the modal data calculated from the FE dynamic analysis for the initial reduced-order model, are summarised in Tables 1 and 2.

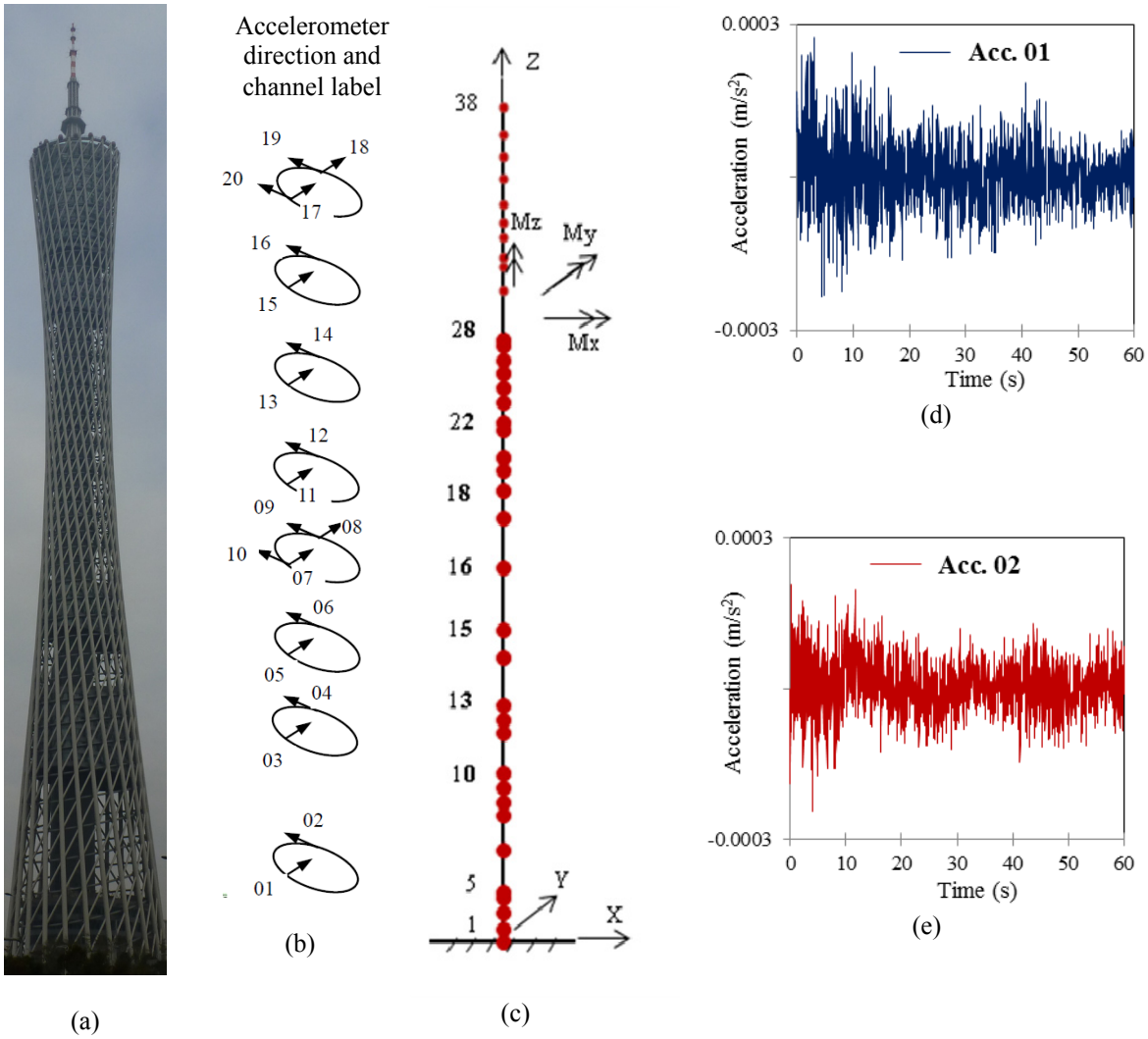

Figure 1. The Canton Tower (a) Actual structure; (b) Positions of installed accelerometers; (c) Reduced FE model; (d) and (e) Measured accelerations of accelerometers 01 and 02

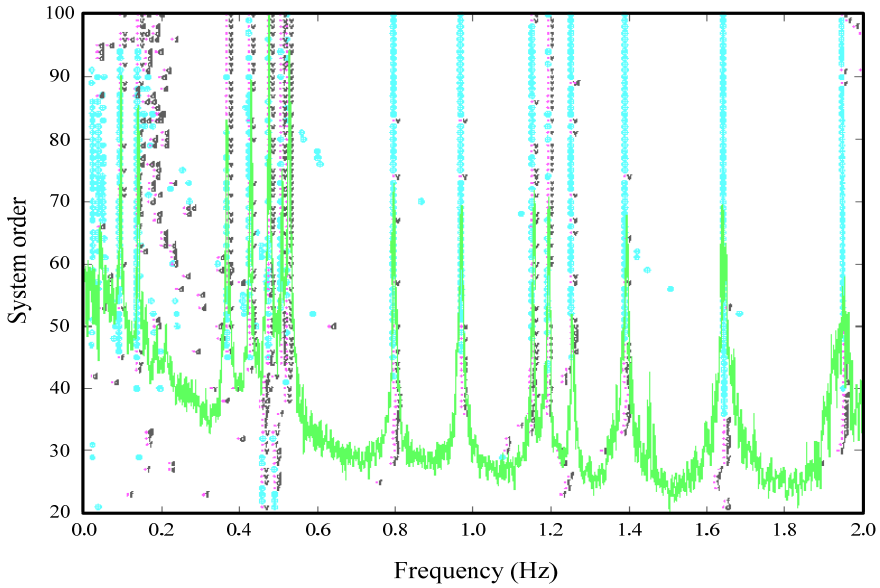

Figure 2. The stabilization diagram of measured acceleration data used for operational modal analysis

The results in the tables show that the difference between the measured and numerical frequencies is relatively large, with the average frequency absolute errors of 12.976% and 10.800% for first 5 and 11 frequencies, respectively. The MAC diagonal values, calculated from the identified incomplete mode shapes and the numerical eigenvectors restricting to the same DOFs, indicate reasonable correlation between the identified and numerical modes, except the 6th mode (i.e. the first torsion). The significant difference between the measured and numerical frequencies indicates that there could be modelling errors in the reduced numerical model and measurement uncertainty from the recorded vibration data. The proposed method is capable of taking modelling errors and measurement uncertainty into account in the model updating processes so as to provide reliable results for structural updating parameters.

The updated modal properties such as frequencies and MAC diagonal values of the reduced FE model are summarised in Table 1, where the first five measured incomplete modes are utilised for model updating. The results show that the updated frequencies are much closer to the frequencies identified from vibration measurements, reducing the average frequency absolute error from 12.976% initially to 4.721% after updating. The obtained MAC diagonal values indicate that the updated mode shapes correlate well with the initial modes of the numerical model and also have good correlation with the modes identified from field measurements.

Table 1. Initial and updated modal properties of the reduced-order FE model of the Canton Tower using five field measured incomplete modes

Mode	FE frequency (Hz)	Tested frequency (Hz)	Initial error (%)	MAC initial, tested	Updated frequency (Hz)	Updated error (%)	MAC updated, initial	MAC updated, tested
1	0.1109	0.094	17.995	0.9004	0.0914	-2.752	0.9081	0.9846
2	0.1588	0.140	13.458	0.9341	0.1378	-1.566	0.8599	0.9875
3	0.3473	0.367	-5.371	0.9141	0.3655	-0.408	0.9820	0.8991
4	0.3691	0.421	-12.332	0.9438	0.3879	-7.862	0.9856	0.9609
5	0.4003	0.475	-15.724	0.8811	0.4227	-11.016	0.9887	0.8689
Average absolute error (%)			12.976			4.721		

In order to investigate the influence of the modal information required on the performance of structural model updating, information about first 10 measured incomplete modes (excluding the 6th mode) is adopted for updating the reduced FE numerical model, as shown in Table 2. Again, the updated frequencies become closer to the corresponding measured frequencies, with the average frequency absolute errors reduced from 10.800% to 5.124%. The obtained high MAC diagonal values indicate that the updated mode shapes correlate well with both the initial numerical eigenvectors and measured mode shapes.

Table 2. Initial and updated modal properties of the reduced-order FE model of the Canton Tower using 10 field measured incomplete modes

Mode	FE frequency (Hz)	Tested frequency (Hz)	Initial error (%)	MAC initial, tested	Updated frequency (Hz)	Updated error (%)	MAC updated, initial	MAC updated, tested
1	0.1109	0.094	17.995	0.9004	0.0951	1.204	0.9306	0.9603
2	0.1588	0.140	13.458	0.9341	0.1401	0.078	0.8971	0.9747
3	0.3473	0.367	-5.371	0.9141	0.3635	-0.961	0.9832	0.8689
4	0.3691	0.421	-12.332	0.9438	0.4132	-1.853	0.9977	0.9536
5	0.4003	0.475	-15.724	0.8811	0.4367	-8.064	0.9944	0.8519
6	0.4617	0.505	-8.575	/	0.4614	-8.626	0.7506	/
7	0.4868	0.519	-6.207	0.7691	0.4968	-4.280	0.9712	0.8696
8	0.7384	0.789	-6.419	0.7792	0.7776	-1.445	0.9881	0.8206
9	0.9037	0.967	-6.544	0.7717	0.9301	-3.820	0.9825	0.8477
10	0.9973	1.151	-13.355	0.7015	0.9932	-13.714	0.9889	0.6776
11	1.0375	1.190	-12.818	0.7519	1.0434	-12.321	0.9889	0.8203
Average absolute error (%)			10.800			5.124		

The results for the updated stiffness parameters of the numerical model are shown in Figure 3, where the two cases with 5 and 10 measured incomplete modes used for model updating are considered. The values of stiffness updating parameters are relatively small and reasonably consistent for both cases. In the case with 5 measured modes used in model updating, the obtained stiffness updating parameters range from -8.77% to 8.24% with an average of the absolute values of 2.70%. For the case with 10 measured modes used, similar results are obtained and the updated stiffness parameters vary between -10.08% and 11.36% with an average of the absolute values of 3.41%. The results for the adjusted stiffness parameters for both cases are very close to each other. These results appear reasonable in practice since only relatively small stiffness modifications in the well-constructed initial FE model are required to minimise significant differences between the numerical and identified modal data.

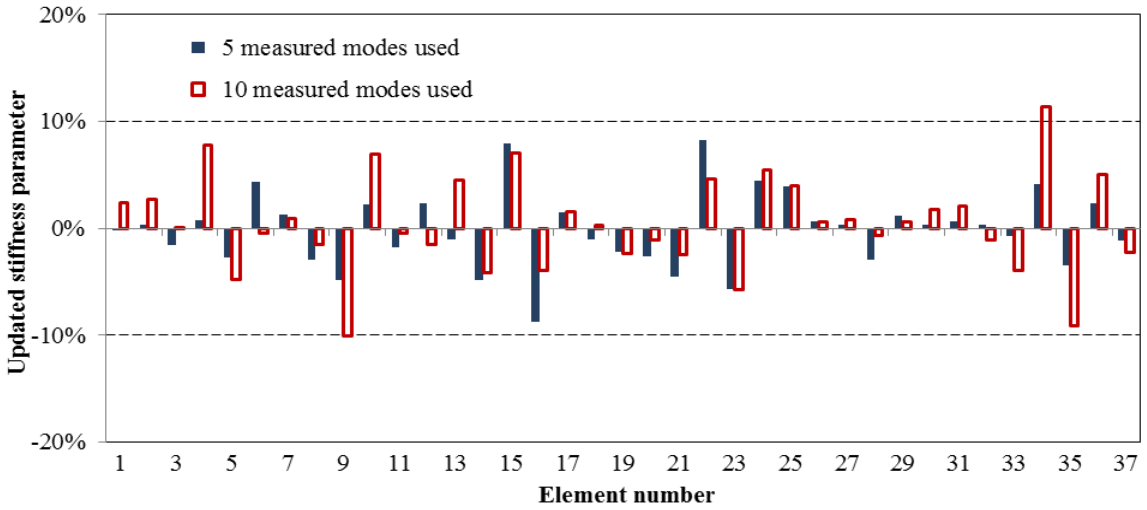

Figure 3. Updated stiffness parameters of the FE numerical model of the Canton Tower, five and 10 measured incomplete modes used

CONCLUSIONS

An effective model updating method is proposed for adjusting FE models of structural dynamic systems, requiring only limited information about measured incomplete mode shapes. The proposed dynamic perturbation method gives the exact relationship between the perturbation of structural parameters and the modal properties of the actual tested dynamic structure. Structural updating parameters are chosen to characterise the difference in structural parameters between the numerical model and tested structure at element level. From the obtained exact relationship, an iterative solution algorithm is developed to solve for the chosen structural updating parameters in the least squares sense. The regularization algorithm based on the Tikhonov solution incorporating the L-curve criterion is employed to filter out the influence of modal data measurement uncertainty and then to produce reliable solutions for the updating parameters. Finally, the practical applications of the proposed method are demonstrated by correctly updating the Canton Tower benchmark problem using the operational modal properties identified from ambient vibration measurements.

Based on the results from practical applications, the following conclusions are drawn. 1) The proposed approach needs much less computational effort to evaluate the structural updating parameters, and provides optimised solutions for model updating in the least squares sense without requiring optimisation techniques. 2) The proposed approach performs well for the Canton Tower benchmark problem considered and gives reliable predictions of structural updating parameters even in the cases where relatively large modifications in structural parameters and/or modal properties exist between the numerical model and tested structure. 3) A regularization algorithm, the Tikhonov regularization method incorporating the L-curve criterion, is introduced to give stable solutions for the structural updating parameters by reducing the influence of measurement errors in modal data. 4) Finally, the proposed model updating method provides reasonably small modifications of structural properties to minimise the relatively large difference between the modal data calculated from the initial FE model and those identified from the vibration measurements.

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