Inverse analysis of rebar stress in reinforced concrete beams with crack opening displacement

Takashi Matsumoto, Keita Sato, Toshiro Hayashikawa, Xingwen He
Hokkaido University, Sapporo, Japan

ABSTRACT: This paper presents the problem formulation of the forward and inverse analysis of a reinforced concrete fracture and conducts the inverse analysis of ten crack opening displacement (COD) data sets in order to estimate rebar stress and diameter. Inverse analysis is carried out for ten noisy COD data sets with various load levels and rebar diameters. Two methods to determine rebar diameter are examined and compared. The estimate results of ten data sets shows that the error decreases as the load level increases or the rebar diameter increases. This can be because, in these conditions, the rebar takes more force and contributes more to crack closure.

1 INTRODUCTION

Infrastructure aging problem is becoming an important issue in a country that experienced a mass construction period in the past. In order to manage the coming mass maintenance period, it is necessary to develop a high resolution inspection method that gives further quantitative understanding of structural degradation and load capacity towards the successful structural health monitoring.

In the case of reinforced concrete structures, it is crucial to have enough cross sectional area of steel rebars throughout service life, especially under corrosive environment. In order to check the rebar conditions, destructive testing methods, such as cover concrete removal, can be employed to directly observe the rebar cross sectional area, or non destructive testing methods, such as electrochemical methods, can be carried out to measure the corrosion susceptibility of rebars, thereby indirectly evaluating their cross sectional area.

This paper studies the estimation of rebar stress and diameter in a reinforced concrete beam based on exterior crack opening displacement measurement (COD). The relation between crack opening displacement and bridging stress transmitted across a crack is modeled based on fracture mechanics, and this problem is solved inversely to obtain the transmitted bridging stress distribution on the crack surface from the measured crack opening displacement profile.

The problem formulation of the forward and inverse problem is described briefly. It is shown that the rebar stress and location can be obtained with the inverse analysis, and that the rebar diameter can be estimated with two proposed methods. This shows that, if further developed, the current method can be used as a non destructive testing method in order to evaluate the interior rebar stress based on the exterior crack opening displacement.

2 PROBLEM FORMULATION

The model of a reinforced concrete beam under flexure is described below. It is assumed that concrete and steel are a linear elastic material and that, under flexural loading, a single crack initiates and develops in the center on the tension face. This crack is assumed as a Mode I crack, where only normal bridging stress is transmitted across the crack.

Bridging stress is exerted by concrete aggregates and steel rebars. In the current study, the aggregate bridging is ignored, and only the bridging exerted by rebars is taken into account, since the effect of aggregate bridging is negligible compared to the crack tip shielding effect of rebars. The current model is schematically shown in Figure 1.

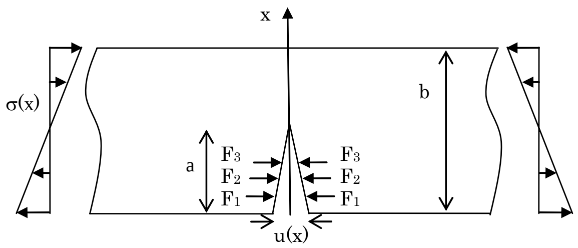

Figure 1. Reinforced concrete beam under flexure. Rebars are bridging the crack with forces, F_i.

Crack opening displacement of a reinforced concrete beam, *u'*, can be related to the bridging stress and the externally applied load through the following equation:

$$u'(x) = \frac{4}{E'} \int_x^a \int_0^{a'} G(x', a', b)[\sigma(x') - f(x')]dx' G(x, a', b)da' \tag{1}$$

where $E' = E_c$ for plane stress and $E' = E_c / (1 - \nu^2)$ for plane strain respectively, E_c the Young's modulus of concrete, and ν the Poisson's ratio. $\sigma(x)$ is the stress from external flexural loading that would exist in the crack line if there was no crack (shown linear in Figure 1 at the edges), and $G(x, a, b)$ is the weight function to determine stress intensity factors, where a is the crack length and b the beam height. Weight functions of various configurations including the current configuration can be found in the stress intensity factor handbook (Tada et al. (1985)), and the weight function of the current configuration can be found elsewhere (Cox and Marshall (1991); Islam and Matsumoto (2003)). $f(x)$ is the bridging stress distribution of rebars, and is expressed with the use of a Heaviside step function:

$$f(x) = \sum_{i=1}^{r} f_i \left[H(x - h_i) - H(x - h_i - d_b)\right] \tag{2}$$

where $f_i = (F_i / d_b)$ is the uniform rebar stress within its diameter along the crack length, F_i the total force in i-th layer, r the total number of reinforcement layers, h_i the distance of a layer from the bottom face, d_b the bar diameter.

With the externally applied load given and F_i obtained via section analysis, the forward problem of Eq. (1) can be numerically approximated and solved (Islam and Matsumoto (2003); Islam and Matsumoto (2004)).

The inverse problem is to determine the distribution of $f(x)$ along the crack length from the measured $u(x)$. It is not necessary to know the function form of Eq. (2), but to know the crack opening displacement profile, $u(x)$, at a prescribed load level. Such an inverse problem is ill-posed, therefore Tikhonov regularization method is applied to solve the problem with noisy COD data.

The net crack opening displacement, *u'*, is the sum of opening due to externally applied loading, $\bar{u}$, and closing due to rebars, *u*.

$$u(x) = \frac{4}{E'} \int_x^a \left[\int_0^{a'} G(x', a', b) f(x') dx' \right] G(x, a', b) da' = \bar{u}(x) - u'(x) \tag{3}$$

We consider Eq. (3) as a linear operator equation

$$Tf = u \text{, with } f \in Z \text{ and } u \in U \tag{4}$$

with

$$T : Z \to U \, . \tag{5}$$

F and *U* are infinite dimensional real Hilbert spaces with corresponding inner products and norms.

We are interested in the approximate solution giving the extremals of the following Tikhonov functional

$$M^{\alpha}[f] = \|T_h f - u_{\delta}\|_U^2 + \alpha \|f\|_Z^2 \tag{6}$$

where T_h is the numerical approximation of the transform, *T*, and $\alpha > 0$ the regularization parameter. We have $u_\delta \in U$, the available noisy data up to noise level δ. After finite difference approximation, we reach at the following normal equation:

$$\mathbf{B}\, f + \alpha\, \mathbf{C}\, f = \mathbf{v} \, . \tag{7}$$

Detail theoretical assumptions and numerical method for solving the inverse problem are available in Islam and Matsumoto (2004).

3 ESTIMATE OF REBAR STRESS AND DIAMETER

The reinforced concrete beam model analyzed in this study (Figure 1) is 10 cm wide, 10 cm high, and 40 cm long. It is assumed that there is only one layer of rebars with the cover of 3.2 cm.

Inverse analysis is carried out with ten COD data sets. These data sets are made by adding normally distributed noise to COD data obtained from forward analysis. The average of normally distributed noise is zero, and the variance is set to be 3 % of the maximum COD value. Furthermore, this noise-added data is smoothed with polynomial approximation that is obtained by least square fit. An example of smoothed noisy COD data is shown in Figure 2, and the rebar stress distribution obtained by inverse analysis from the COD data in Figure 2 is shown in Figure 3.

According to the manner described above, the ten COD data sets are prepared for various load levels and rebar diameters. The load levels are below the yield strength of rebar, and there are three diameters cases: 5, 6, and 7 mm. These ten cases are shown in Table 1.

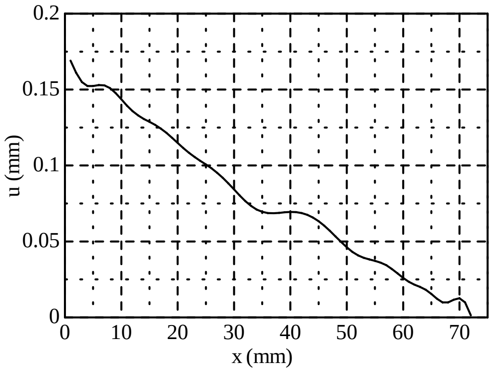

Figure 2. Smoothed noisy COD data.

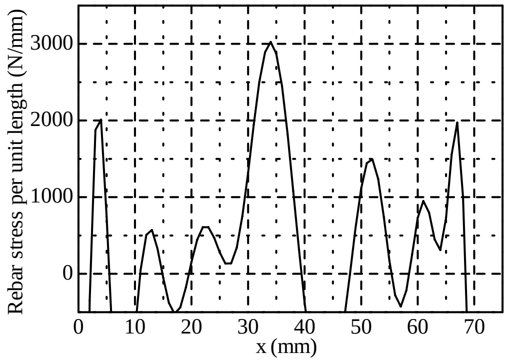

Figure 3. Rebar stress obtained by inverse analysis.

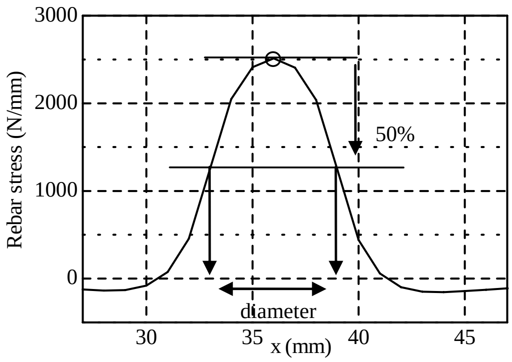

Figure 4. Estimate of diameter by half height.

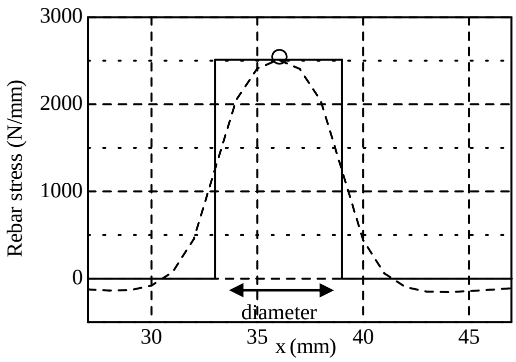

Figure 5. Estimate of diameter by equivalent area.

For the estimate of rebar stress and diameter, it is necessary to have a procedure to determine those values. Although Figure 3 shows the estimated rebar stress distribution along the crack, in reality, the stress exerted by rebars should not be distributed. It is interpreted that the rebars are located around 35 mm where the peak bridging stress is observed. It is also seen that other minor peaks are obtained together. These peaks are due to the noise of COD data and should not be used for the estimate of rebar stress and diameter.

In the current study, two methods are examined. One is the procedure that the diameter is determined by the distance between two points at the 50 % of the peak stress value (Figure 4). Another one is that the diameter is determined so that the area below the stress distribution curve and above zero can be equivalent to that of a rectangle with the peak stress value and the diameter (Figure 5). With the diameter obtained by each method, rebar stress is calculated by the product of the diameter and the peak stress.

Table 1 shows the results on the estimated rebar diameter obtained by two methods. The error is calculated with the given diameter. The maximum error is 17 %, and the minimum 0.05 %. It is found that the error decreases as the load level increases or as the diameter increases. This can be because, in these conditions, the rebar takes more force and contributes more to crack closure. The difference between the two methods is not so significant, but the half height method gives slightly better estimate. Therefore, the following estimate of rebar stress is carried out only by half height method.

Table 2 shows the estimated rebar stress and the theoretical rebar stress. The error is also calculated. The theoretical stress is calculated based on section analysis:

Table 1. Estimated rebar diameters by two methods.

Case	Rebar diameter (mm)	Total load (kN)	Rebar diameter by half height method (mm)	Error (%)	Rebar diameter by equivalent area method (mm)	Error (%)
1	5	16	5.71019	14.20	5.8502	17.00
2	5	20	5.45433	9.09	5.61258	12.25
3	5	22	5.53538	10.71	5.65182	13.04
4	6	16	6.44124	7.35	6.31095	5.18
5	6	20	6.12308	2.05	6.23816	3.97
6	6	24	5.92717	1.21	6.0485	0.81
7	7	16	6.77901	3.16	6.70543	4.21
8	7	20	7.32832	4.69	7.44305	6.33
9	7	24	7.14662	2.09	7.19459	2.78
10	7	28	6.99658	0.05%	7.1134	1.62%

Table 2. Estimated rebar stress and theoretical rebar stress.

Case	Theoretical rebar stress (MPa)	Estimated rebar stress (MPa)	Error (%)
1	272.4020	196.9448	27.70
2	340.3217	284.3857	16.44
3	374.2815	319.2877	14.69
4	193.7623	158.6318	18.13
5	242.0747	216.8208	10.43
6	290.3870	302.9918	4.34
7	145.7197	180.8180	24.09
8	182.0538	182.4018	0.19
9	218.3866	233.3590	6.86
10	254.7207	243.3388	4.47

$$f_{th} = \frac{M}{A_s jd} \tag{8}$$

where M is external moment, A_s is rebar section area, d is effective depth, and jd is arm length. Again, it is found that the error decreases as the load level increases or the diameter increases.

4 CONCLUSIONS

This paper presented the problem formulation of the forward and inverse analysis of a reinforced concrete fracture and conducted the inverse analysis of ten crack opening displacement (COD) data sets in order to estimate rebar stress and diameter.

The forward problem of a reinforced concrete fracture was formulated. In the forward problem, crack opening displacement is solved with externally applied load and rebar forces given. On the other hand, the inverse problem was to solve for rebar forces with the crack opening displacement and the externally applied load given. The inverse problem is ill posed due to the noise of crack opening displacement. Therefore, the problem is regularized and numerically solved.

Inverse analysis was carried out with ten COD data sets for a small reinforced concrete beam. The COD data was made by adding normally distributed noise to the COD obtained by forward analysis. The ten data sets were prepared for various load levels and rebar diameters.

Two methods to determine the rebar diameter were examined. A method which determines the diameter with the distance between two points at the 50 % of the peak stress value showed slightly better estimate than another method.

The estimate results of ten data sets showed that the error decreases as the load level increases or the rebar diameter increases. This can be because, in these conditions, the rebar takes more force and contributes more to crack closure.

For further development as one of SHM technologies, a systematic experimental program should be conducted where various load levels and rebar diameters are included. Also, it is necessary to develop a high resolution image acquisition system for the usage on site.

References

Cox, BN and Marshall, DB. 1991. Stable and unstable solutions for bridged crack in various specimens. *Acta Metallurgica et Materialia*, 39(4):579-589.

Islam, MN and Matsumoto, T. 2003. Determination of steel stresses in reinforced concrete structures from crack opening profile. Proc. of the First Int. Conf. on Structural Health Monitoring and Intelligent Infrastructure, Tokyo, 739-746.

Islam, MN, and Matsumoto, T. 2004. Inverse analysis to determine re-bars' force from external crack widths measurement. *Journal of Applied Mechanics*, JSCE, 7(2): 1179-1186.

Tada, H, Paris, PC, and Irwin, GR. 1985. *The stress analysis of cracks handbook*, St. Louis: Paris Productions Inc.