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Modal Parameter Estimation Via Padé Approximants

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ABSTRACT: The output-only identification of modal parameters of large structures with low damping and under adverse signal-to-noise (SNR) condition is a practical problem of interest that still poses an enormous challenge since usual modal identification techniques generally prove inaccurate, particularly when damping estimation is concerned. This work addresses this problem by presenting a novel estimation technique based on Padé approximants, derived from quantum mechanics resonance problems. Via Monte Carlo simulation techniques we show the superior performance of the new approach, illustrating some of its statistical behavior. Finally, we apply it to a well known real structure.

1 INTRODUCTION

Modal analysis of large civil structures has well known relevance for structural design, damage identification and diagnosis. To achieve high identification accuracy, one must often resort to the use of expensive techniques for modal excitation, as the problem involves the identification of slowly damped modes that are difficult to characterize in the sole presence of the natural excitation provided by wind or traffic. The difficulty stems from the poor performance of most modal identification methods when the signal-to-noise ratio (SNR) is small. In Coelho & Baccalá (2004) we showed that a state-of-the-art method employing subspace identification can accurately pinpoint modal frequencies but is less accurate in regard to damping coefficient estimation. Furthermore, the latter method was also shown to degrade abruptly around an SNR of 5dB which is not uncommon in practical settings.

To overcome these limitations, we have looked into alternative approaches and the use of Padé approximants, a fairly classical theme, has proved useful in the conditions of natural excitation based identification, which is usually characterized by large amounts available data under low SNR.

The motivation for the present approach comes from the analogous problem in nuclear magnetic resonance (NMR), where frequency and damping rate allow chemical compound identification. NMR identification via Padé approximants was pioneered by Belkic (2005) under comparatively benign conditions where SNR=50dB. Padé approximants were also employed by O´Sullivan & Cowan (2006) to obtain the transient acoustic impulse response of theaters. In this latter case, the fast damping range of the transients facilitated the separation between the actual observed dynamics and the effects of noise as discussed in Sec. 2. The problem of modal analysis, via a semi-empirical approach, has also appeared in the well-founded work of Barone et al. (1998, 2000, 2005).

The rest of the present paper is organized as follows: we begin with a brief revision the fundamentals of Padé approximation and discuss the fundamental role of the idea of Froissart doublets followed in section 3 by our proposed algorithm for the natural excitation setting in which several independent observations are available. Section 4 contains numerical Monte Carlo experiments that illustrate the accuracy and the gentle algorithm degradation with decreasing SNR. In Sec. 5 we show the results of applying the algorithm to the modal identification of the Z24 bridge (Brinker et al. (2001), Maeck et al. (2001)) and briefly state the conclusions in Sec. 6.

2 PADÉ APPROXIMANTS

The general modal response signal $x(t)$ can be written as a sum of *W* damped sinusoids

$$x(t) = \mathrm{Re}\left[\sum_{k=1}^{W} a_k e^{\varphi_k + (d_k + j\omega_k)t}\right] \tag{1}$$

where $a_k > 0$ represent signal amplitudes, $\omega_k \in [-\pi, \pi]$ the modal frequencies, d_k the damping factors and where $\{\varphi_k\}$ are uniformly distributed phases (in $[-\pi, \pi]$). Also, let $j = \sqrt{-1}$ and $t = n.\Delta t$ represent discrete time samples obtained at a sampling rate $f_S = 1/\Delta t$. In practice one must also consider the effect of additive random noise $r(t)$ leading to an observed signal model of the form

$$c(t) = x(t) + r(t) \tag{2}$$

where the noise $r(t)$ is further assumed zero mean Gaussian of σ^2 variance.

Modal identification is based on using an observed finite sequence $f(n)$ of *L* samples of $c(t)$ taken at $t = n.\Delta t$ to determine the parameters in (1).

One may represent the discrete time observations with help of its z-transform written as

$$f(z) = \sum_{n=1}^{L} c_n z^n \tag{3}$$

whose description may be further compacted with help of the so-called Padé approximants which consist of rational functions $[M/N]_f(z) = \dfrac{P_M(z)}{Q_N(z)}$, where *M* and *N* are the degrees of polynomials that approximate the power series expansion (Taylor series) of $f(z)$, an analytical function on $z = 0$, up to degree $M + N$.

Take the power series $f(z) = \sum_{i=0}^{\infty} c_i z^i$

The Padé approximant $[M/N]_f(z) = \dfrac{P_M(z)}{Q_N(z)}$ must satisfy:

$$f(z) - \frac{P_M(z)}{Q_N(z)} = O(z^{M+N+1}) \quad \text{when} \quad z \to 0 \text{ and } Q_N(0) \neq 0,$$

where $O(.)$ is Landau´s big-Oh symbol, meaning that the right side is a power series over z^k, beginning with degree $k = M + N + 1$, up to $k = +\infty$.

As a consequence
$$\sum_{i=0}^{M+N} c_i z^i - \frac{P_M(z)}{Q_N(z)} = 0 \tag{4}$$

We usually impose a normalization condition, $Q_N(0) = 1$, as the ratio $[M/N]_f(z)$ can be multiplied by any number and we must have $Q_N(0) \neq 0$.

So, let
$$P_M(z) = p_0 + p_1 z + \ldots + p_M z^M$$

and
$$Q_N(z) = 1 + q_1 z + \ldots + q_N z^N$$

Multiplying (4) by $Q_N(z)$ results $Q_N(z) \sum_{i=0}^{M+N} c_i z^i - P_M(z) = 0$

From this relation we can extract the following equations:

$$\begin{aligned}
c_0 &= p_0 \\
c_1 + c_0 q_1 &= p_1 \\
c_2 + c_1 q_1 + c_0 q_2 &= p_2 \\
&\vdots \quad\quad \vdots \\
c_M + c_{M-1} q_1 + \cdots + c_0 q_M &= p_M
\end{aligned}$$

$$\begin{aligned}
c_{M+1} + c_M q_1 + \cdots + c_{M-N+1} q_N &= 0 \\
&\vdots \quad\quad \vdots \\
c_{M+N} + c_{M+N-1} q_1 + \cdots + c_M q_N &= 0
\end{aligned}$$

Or, in matrix form:

$$\begin{bmatrix} c_0 & 0 & \cdots & 0 \\ c_1 & c_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ c_M & c_{M-1} & \cdots & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ q_1 \\ \vdots \\ q_M \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_M \end{bmatrix}$$

$$\begin{bmatrix} c_M & c_{M-1} & \cdots & c_{M-N+1} \\ c_{M+1} & c_M & \cdots & c_{M-N+2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{M+N-1} & c_{M+N-2} & \cdots & c_M \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{bmatrix} = - \begin{bmatrix} c_{M+1} \\ c_{M+2} \\ \vdots \\ c_{M+N} \end{bmatrix}$$

For higher values of *M* and *N*, solving for polynomial coefficients is often difficult unless more numerically accurate methods based on singular value decomposition (SVD) are used (Press et al. (1992)).

When one computes $[0/2W]_f$ the resulting rational approximant reduces to the well known Prony method for fitting exponentials to data (Kay (1988)). In the Prony method, the modal parameters $z_k = e^{d_k + j\omega_k}$ are obtained as the roots of $Q_{2W}(z)$. It is also well known that the Prony approach suffers from two serious drawbacks. First of all the 'order' *2W*, i.e. the number of sinusoids, must be known *a priori.* But perhaps, more importantly, the method is very sensitive to additive noise (no less than 80 dB of SNR is adequate), making it useless for the problem at hand.

It may therefore seems surprising that insisting on Padé approximation would lead anywhere. However, more general approximants of the $[N/N]_f$ type do have a peculiar behavior with respect to the effect of noise. This behavior was first noted by Froissart and reported by one of his associates (Bessis (1996)).

In the noise-free case, i.e. when estimation errors are due solely to numerical rounding effects in both computing the approximant and the factorization of the resulting polynomial, the roots of $Q_N(z)$ can be split into two subsets: (a) one containing barely perturbed estimates of z_k (true poles) and another subset of non-modal roots that are circularly distributed at a constant radius from the origin of the complex plane (spurious poles). Furthermore a subset of the roots of $P_N(z)$ are in close correspondence with the non-modal roots by forming the so called Froissart doublets, i.e. close pole-zero root pairs from $Q_N(z)$ and $P_N(z)$ (respectively), whose distance is of the order of the numerical machine precision used in the computations (See Fig. 1 and Fig. 2). The proximity of the non-modal poles and zeros is easy to understand as the poles need numerator zeros to approximately cancel their effect on $f(z)$.

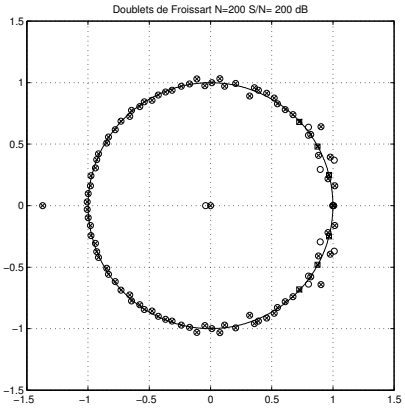

Figure 1. Froissart doublets clustered near unit circle (L=200, SNR=200dB)

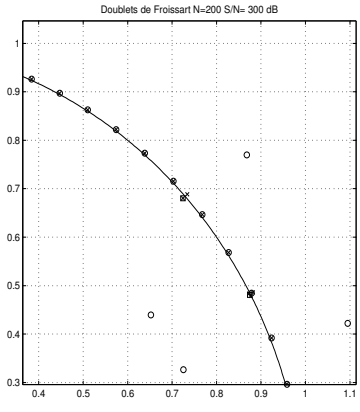

Figure 2. Close-up view showing pole(x)-zero(o) cancellation (under the conditions of Fig. 1)

Another remarkable feature of the non-modal (spurious) roots of $Q_N(z)$ is that the Cauchy residues associated to them are usually much smaller than the residues associated with the modal roots. Both this geometrical pattern of Froissart doublets and comparative residue size carry over to when there is substantial additive noise and these properties can be used as separation criteria between modal and non-modal roots for $Q_N(z)$ (see Fig. 3).

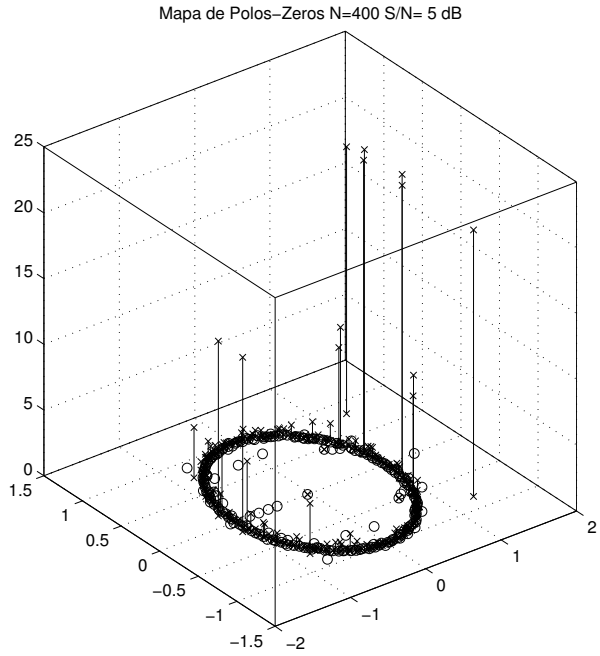

Figure 3. Cauchy residues of modal and non-modal (spurious) roots of $Q_N(z)$

3 PROPOSED ALGORITHM

The algorithm begins by computing the signal autocorrelation function $C(\tau)$ for a given realization of $c(t)$, comprising L signal samples taken at $t = n.\Delta t$, $\Delta t = 1/f_S$ In the next step an $[L/2/L/2]_C$ Padé approximant is computed using the z-transform equivalent of $C(\tau)$ for positive τ.

The next step consists of Froissart doublet identification on the resulting approximant and is done by matching each pole to its closest zero. The distance obtained in this pairing, together with the Cauchy residue of each pole are used as a vector input for the final step of the algorithm consisting of classification by clustering.

For clustering we employ the k-means algorithm (Abonyi & Feil (2007)). Each pole is classified according to its distance to the centroid of a class, where modal poles ideally have large residues and large pole-zero distance. As the vector is incorporated to an identified nearest class, the centroid is updated and the process proceeds until the last pole has been classified. We operate with two classes, hoping to isolate true modal poles from spurious ones.

A typical distribution of points over the clustering identification space is shown on Fig. 4 for a single signal realization. Two classes were defined and their centroids are indicated by small circles. The farthest points from the origin are candidates to be identified as true modal poles. The algorithm can produce both false positive and false negative results. To mitigate this occurrence, the results of several independent, i.e. non-overlapping realizations, can be used.

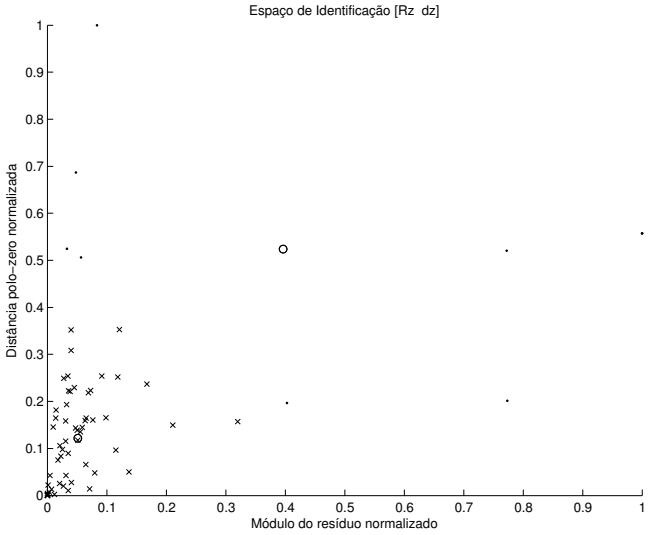

Figure 4. Separation of poles into two classes: modal poles (solid dots) and spurious poles (small circles). The large circles indicate the centroids of two classes.

4 NUMERICAL EXPERIMENTS

To give an idea of the performance attainable with the present procedure, we simulated realizations of a synthetic signal formed by the sum of three dumped sinusoids with the following parameters: sampling frequency $f_S = 25Hz$, frequencies $\omega_1 = 2\pi.1\ rad/s$, $\omega_2 = 2\pi.2\ rad/s$ and $\omega_3 = 2\pi.3\ rad/s$, damping $d_1 = 1\%$, $d_2 = 2{,}5\%$ and $d_3 = 5\%$ and amplitudes $a_1 = 0{,}8$, $a_2 = 0{,}9$ and $a_3 = 1$. The noise power for each test was modified, varying the signal to noise ratio (SNR) from -10dB to 30dB at 5 dB steps. Also the sample length *L* was incremented by 2, ranging from 40 to 120 sample points. The final position of the estimated poles was obtained by averaging the estimates of the poles over 200 realizations.

The mean squared error (MSE) of pole position for the 1Hz sinusoid is shown on Fig. 5 where one can see a periodic "valley" structure with better performance around certain particular values of *L*. This behavior is the subject of current research. Also noticeable is the gentle decrease in performance as SNR decreases. As evidenced by Fig. 6, bias of the estimates are also small and start to collapse around 0dB for certain sample lengths in a remarkable performance given the low SNR conditions of the test.

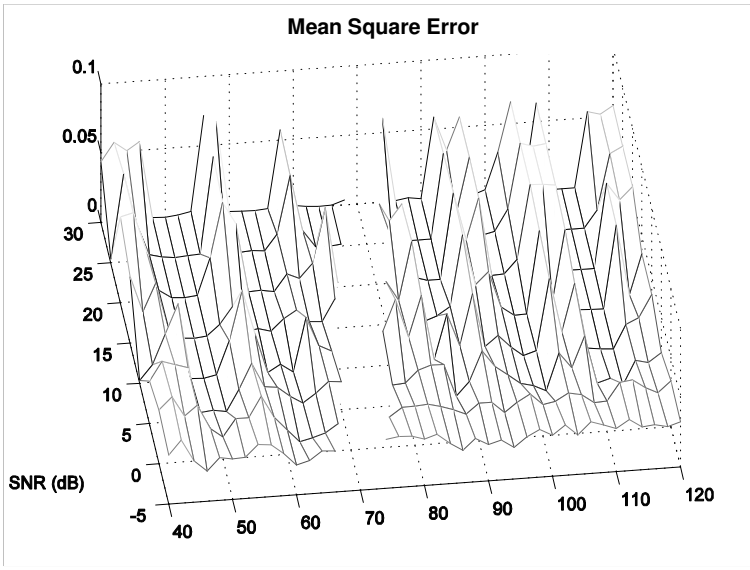

Figure 5. Mean squared error for the 1 Hz sinusoid identification using the proposed algorithm.

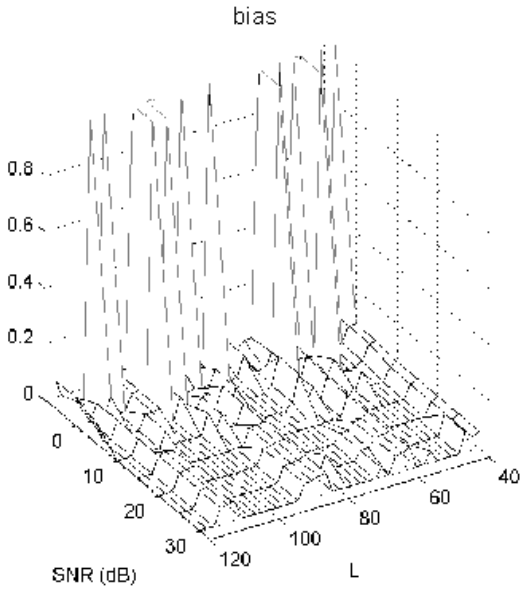

Figure 6. Bias of the 1 Hz sinusoid identification by the proposed algorithm.

5 MODAL IDENTIFICATION OF A REAL STRUCTURE

Table 1 portrays the comparatively good agreement of the present algorithm with other modal identification techniques applied to Z24 bridge data. For more information about this reference structure, see http://www.kuleuven.be/bwm/Z24/index.html.

Table 1. First five mode frequencies of Z24 bridge, identified by different techniques.

Reference	Technique	Mode 1 (Hz)	Mode 2 (Hz)	Mode 3 (Hz)	Mode 4 (Hz)	Mode 5 (Hz)
Coelho & Baccalá (2004)	ESPRIT	3,846	4,995	9,762	10,822	-----
Peeters et al. (2001)	FDD	3,856	4,886	9,783	10,31	12,50
Maeck et al. (2001)	SSI	3,84	4,67	9,69	10,14	12,11
This paper	PADÉ	3,864	4,798	9,690	10,317	12,500

6 CONCLUSIONS

In this paper we have shown that Padé approximants may be used for accurate modal identification because the effect of additive noise is confined to an annulus composed of Froissart pole-zero doublets close to the unit circle of the complex plane. Due to low damping, the poles of interest that reflect the modal structure of the signal lie close to the unit circle in our problem. Even though, we show by numerical experimentation that the modal poles may be statistically separated from the spurious poles by discriminating the Cauchy residues of the true poles, when many signal realizations are available.

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