A substructuring method for calculation of eigenvalue derivatives and eigenvector derivatives

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ABSTRACT: A substructuring method is proposed in the paper to calculate the eigenvalue derivatives and eigenvector derivatives of structures. The advantage of the new technique lies in that only the eigensensitivities of the substructures containing the design parameters of interest are calculated while eigensensitivities of other substructures are not. The eigensensitivities of the global structure are assembled by performing some constraints on the eigensensitivities of the substructures at the interface. Consequently this can reduce the computation load significantly especially for model updating in which eigensolutions and eigensensitivities need to be calculated iteratively. Applications to a frame and a practical bridge structure show that the present substructuring approach is more computational efficient than the traditional global method.

1 INTRODUCTION

The numerical results from a finite element (FE) model often differ from the experimental results of real structures. FE model updating is often required to identify and correct the uncertain parameters of FE model and is usually posed as an optimization process. During the sensitivity-based model updating process, the eigensolutions and the eigensensitivity of the analytical model need to be calculated in each iteration (Brownjohn et al, 2001). Although the development of the FE method, together with the growing computational speed and storage capabilities, has made it possible to solve larger and more complex problems, the sensitivity-based model updating of large-scale structures are usually time-consuming or even prohibited.

The eigensensitivity is used to indicate the variation trends of the design parameters. Nevertheless, the cost of calculating the eigensensitivity is always the dominant contributor to the total cost in many model updating procedures. Fox and Kapoor (1968) proposed a modal method to determine the eigenvalue and eigenvector derivatives. Such procedure is sometimes computationally expensive, since it takes account of all modes of the system to calculate the required eigenvalue and eigenvector derivatives. Nelson (1976) proposed a simpler technique in which calculation of eigenvector derivatives of one mode requires the modal parameters of that mode only. Many researchers strive to accelerate the calculation of eigensensitivity, either by reducing the required modes (Wang, 1991, Alvin, 1997), or by reducing the degree of freedom (DOF) of the structure (Lin, 1995). However, most of them require to recalculate the eigensensitivity in the global structure level, even if only one design parameter is changed. This paper intends to accelerate the calculation of eigensensitivity by considering only particular substructures instead of the global structure.
Weng and Xia (2007, 2008) proposed a modal truncation technique to improve the efficiency of the Kron’s substructuring method to obtain the eigensolutions. Only some lower modes of the substructures are retained in the technique, while the higher modes are discarded and compensated with the first-order residual flexibility. This paper extends this substructuring technique to calculate the eigensensitivity. The derivative matrices of particular substructures are calculated to assemble the eigensensitivity formula of the global structure. Since the substructures are always much smaller than the global structure, the computation efficiency is improved significantly. The derived formula is applied into a frame structure and a highway bridge, to verify the effectiveness and accuracy of the proposed method.

2 BASIC THEORY

The substructuring method for eigensolutions calculation mainly includes three steps (Kron, 1963): first, the global structure with \( N \) DOFs is divided into \( NS \) substructures according to some criteria; second, each substructure is analyzed independently. For example, the \( j \)th substructure has \( n(j) \) DOFs \((j = 1, 2, \ldots, NS)\) and \( n(j) \) eigenpairs as: \( \Lambda_j = \text{Diag} [\lambda_1(j), \lambda_2(j), \ldots, \lambda_{n(j)}(j)] \), \( \Phi_j = \begin{bmatrix} \phi_1(j), & \phi_2(j), & \ldots, & \phi_{n(j)}(j) \end{bmatrix} \); third, the divided substructures are reconnected based on virtual work and geometric compatibility to recover the global structure. Due to the orthogonality properties, the eigenequation of the assembled global structure is transformed into (Sehmi, 1989):

\[
\begin{bmatrix}
\Lambda^p & -\Gamma \\
-\Gamma^T & 0
\end{bmatrix}
\begin{bmatrix}
z \\
\tau
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
z \\
\tau
\end{bmatrix},
\Gamma = [C\Phi^T]^	au
\]

In this equation, \( \Lambda^p = \text{Diag}[\Lambda^{(1)}, \Lambda^{(2)}, \ldots, \Lambda^{(NS)}] \), \( \Phi^p = \text{Diag}[\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(NS)}] \), and \( C \) is a rectangular connection matrix, which constrains the interface DOFs in adjacent substructures. \( \tau \) is the internal connection forces, and \( \bar{\lambda}_i \) is the \( i \)th eigenvalue of the global structure. \( z \) is the mode participation factor, and indicates the contribution of each mode of the substructures to the modes of the global structure by \( \bar{\Phi} = \Phi^p \{z\} \).

In Kron’s substructuring method, the complete eigensolutions of all substructures are required to assemble the primitive form of \( \Lambda^p \) and \( \Phi^p \). Weng and Xia (2007, 2008) proposed the first-order residual flexibility based substructuring (FRFS) method to improve the calculation of eigensolutions in terms of both accuracy and efficiency. If there are \( m(j) \) eigensolutions in the \( j \)th substructure retained as ‘master’ modes. The residual higher modes are discarded as ‘slave’ variables and compensated by the first-order residual flexibility. For the \( i \)th mode, the eigenequation has (Weng and Xia 2007; 2008):

\[
\begin{bmatrix}
\Lambda^m & \Gamma_m \left( \Phi^m \right)^{-1} & \Gamma_m \\
\Gamma_m^T & \gamma_i & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
z \\
\tau
\end{bmatrix} = \bar{\lambda}_i \begin{bmatrix}
z \\
\tau
\end{bmatrix},
\Gamma_m = [C\Phi^m]^T, \quad \gamma_i = C\Phi^m (\Phi^m)^{-1} \left[ \Phi^m \right]^T C^T
\]

In this equation, the subscripts ‘m’ and ‘s’ represent the master and slave variables respectively. \( \Lambda^m \) and \( \Phi^m \) are diagonal assembly of master eigenvalues and eigenvectors of all substructures. \( \Phi^m (\Phi^m)^{-1} \left[ \Phi^m \right]^T \) is the first-order residual flexibility, which can be represented by diagonally assembling the residual flexibility of all substructures. The size of the reduced eigenequation (Eq. (2)) equals to the number of master modes \( m^p = \sum_{j=1}^{NS} m^{(j)} \), which is much smaller than that of the original eigenequation (Eq. (1)). Based on the reduced eigenequation, the eigenvalue \( \bar{\lambda}_i \) and
mode participation factor \( \{ z_i \} \) can be obtained with traditional method, such as Lanczos method (Parlett, 1980) or Subspace Iteration method (Bathe, 1976).

3 EIGENVALUE AND EIGENVECTOR DERIVATIVES

To extend this simplified eigenvalue equation for the eigensensitivity analysis, Eq. (2) is differentiated with respect to an elemental parameter \( r \) (for example flexural rigidity) as:

\[
\frac{\partial}{\partial r} \left[ \begin{array}{c}
\Lambda_m^i + \Gamma_m \left( \Gamma_m^i \left( \Lambda_m^i \right)^{-1} \Gamma_m^i \right)^{-1} \Gamma_m^T - \lambda_m^i \end{array} \right] = \frac{\partial}{\partial r} \left[ \begin{array}{c}
\Lambda_m^i + \Gamma_m \left( \Gamma_m^i \left( \Lambda_m^i \right)^{-1} \Gamma_m^i \right)^{-1} \Gamma_m^T - \lambda_m^i \end{array} \right] \{ z_i \} = \{ 0 \} \quad (3)
\]

Pre-multiplying \( \{ z_i \}^T \) on both sides of Eq. (3) and rearranging the equation, the derivative of eigenvalue \( \lambda_m^i \) with respect to the designed parameter \( r \) is:

\[
\frac{\partial \lambda_m^i}{\partial r} = \{ z_i \}^T \frac{\partial}{\partial r} \left[ \begin{array}{c}
\Lambda_m^i + \Gamma_m \left( \Gamma_m^i \left( \Lambda_m^i \right)^{-1} \Gamma_m^i \right)^{-1} \Gamma_m^T 
\end{array} \right] \{ z_i \} \quad (4a)
\]

In this equation,

\[
\frac{\partial}{\partial r} \left[ \begin{array}{c}
\Lambda_m^i + \Gamma_m \left( \Gamma_m^i \left( \Lambda_m^i \right)^{-1} \Gamma_m^i \right)^{-1} \Gamma_m^T 
\end{array} \right] = \frac{\partial \Lambda_m^i}{\partial r} + \frac{\partial \Gamma_m}{\partial r} \left( \Gamma_m^i \left( \Lambda_m^i \right)^{-1} \Gamma_m^i \right)^{-1} \Gamma_m^T + \Gamma_m \left( \Gamma_m^i \left( \Lambda_m^i \right)^{-1} \Gamma_m^i \right)^{-1} \frac{\partial \Gamma_m}{\partial r} \left( \Gamma_m^i \left( \Lambda_m^i \right)^{-1} \Gamma_m^i \right)^{-1} \left( \Gamma_m^i \left( \Lambda_m^i \right)^{-1} \Gamma_m^i \right)^{-1}
\]

\[
= \frac{\partial \Lambda_m^i}{\partial r} + \frac{\partial \Gamma_m}{\partial r} \left( \Gamma_m^i \left( \Lambda_m^i \right)^{-1} \Gamma_m^i \right)^{-1} \Gamma_m^T + \Gamma_m \left( \Gamma_m^i \left( \Lambda_m^i \right)^{-1} \Gamma_m^i \right)^{-1} \frac{\partial \Gamma_m}{\partial r} \left( \Gamma_m^i \left( \Lambda_m^i \right)^{-1} \Gamma_m^i \right)^{-1} \left( \Gamma_m^i \left( \Lambda_m^i \right)^{-1} \Gamma_m^i \right)^{-1} \quad (4b)
\]

\[
\frac{\partial}{\partial r} \left[ \begin{array}{c}
\left( \Gamma_m^i \left( \Lambda_m^i \right)^{-1} \Gamma_m^i \right)^{-1} 
\end{array} \right] = \left( \Gamma_m^i \left( \Lambda_m^i \right)^{-1} \Gamma_m^i \right)^{-1} \frac{\partial}{\partial r} \left[ \begin{array}{c}
\left( \Gamma_m^i \left( \Lambda_m^i \right)^{-1} \Gamma_m^i \right)^{-1} 
\end{array} \right] \left( \Gamma_m^i \left( \Lambda_m^i \right)^{-1} \Gamma_m^i \right)^{-1}
\]

\[
= C \times \text{Diag} \left[ \frac{\partial}{\partial r} \left[ \begin{array}{c}
\Phi_m^i \left( \Lambda_m^i \right)^{-1} \left( \Phi_m^i \right)^T 
\end{array} \right] \right] \times C^T \quad (4c)
\]

\[
\frac{\partial \Gamma_m}{\partial r} = C \times \text{Diag} \left[ \frac{\partial}{\partial r} \left[ \begin{array}{c}
\Phi_m^i \left( \Lambda_m^i \right)^{-1} 
\end{array} \right] \cdot \Phi_m^i \right] \times C^T \quad (4d)
\]

where \( \frac{\partial \Lambda_m^i}{\partial r} \) and \( \frac{\partial \Phi_m^i}{\partial r} \) are diagonal assembly of the eigenvalue and eigenvector derivatives of the substructures. The eigenvalue and eigenvector derivatives \( \frac{\partial \Lambda_m^i}{\partial r} \) and \( \frac{\partial \Phi_m^i}{\partial r} \) are only calculated in the particular substructure which contains the elemental parameter \( r \), and they are zeros in other substructures.

Since the \( i \)th eigenvector of the global structure can be recovered by:

\[
\Phi_m^i = \Phi_m^i \{ z_i \} \quad (5)
\]

the eigenvector derivative of the \( i \)th mode to the structural parameter \( r \) can be differentiated as:

\[
\frac{\partial \Phi_m^i}{\partial r} = \Phi_m^i \left\{ \frac{\partial z_i}{\partial r} \right\} \quad (6)
\]
So far, once the item \( \frac{\partial \mathbf{z}_i}{\partial r} \) is available, the eigenvector sensitivity of the global structure can be obtained.

Separating the item \( \frac{\partial \mathbf{z}_i}{\partial r} \) into the sum of a particular part and a homogeneous part yields

\[
\frac{\partial \mathbf{z}_i}{\partial r} = \{v_i\} + c_i \{\mathbf{z}_i\}
\]

(7)

where \( c_i \) is a participation factor. Substituting Eq. (7) into Eq. (3), the eigenequation for the \( i \)th mode is transformed into:

\[
\left[ \Lambda_m + \Gamma_m \left( \Lambda_m^T \right)^{-1} \Gamma_m^T \right] \{v_i\} = -\frac{\partial}{\partial r} \left[ \Lambda_m^T + \Gamma_m \left( \Lambda_m^T \right)^{-1} \Gamma_m^T - \mathbf{z}_i \mathbf{1} \right] \{\mathbf{z}_i\}
\]

(8)

and the vector \( \{v_i\} \) can be solved from the above equation (Nelson, 1976).

Regarding the reduced eigenequation Eq. (2), the orthogonal condition of eigenvector satisfies:

\[
\{\mathbf{z}_i\}^T \{\mathbf{z}_i\} = 1
\]

(9)

Differentiating Eq. (9) with respect to \( r \) gives:

\[
\frac{\partial}{\partial r} \{\mathbf{z}_i\}^T \{\mathbf{z}_i\} + \{\mathbf{z}_i\}^T \frac{\partial}{\partial r} \{\mathbf{z}_i\} = 0
\]

(10)

Substituting Eq. (7) into Eq. (10), the participation factor \( c_i \) is obtained as:

\[
c_i = -\frac{1}{2} \left( \{v_i\}^T \{\mathbf{z}_i\} + \{\mathbf{z}_i\}^T \{v_i\} \right)
\]

(11)

Finally, the first-order derivative of \( \{\mathbf{z}_i\} \) with respect to the structural parameter \( r \) is:

\[
\left[ \frac{\partial \mathbf{z}_i}{\partial r} \right] = \{v_i\} - \frac{1}{2} \left( \{v_i\}^T \{\mathbf{z}_i\} + \{\mathbf{z}_i\}^T \{v_i\} \right) \{\mathbf{z}_i\}
\]

(12)

\( \frac{\partial \mathbf{z}_i}{\partial r} \) is derived based on the reduced eigenequation (Eq. (2)), which has a much smaller size than that of the global structure. Consequently, the computation efficiency can be improved significantly. The detailed application of the proposed substructuring method and its efficiency will be discussed in the following examples.

4 NUMERICAL EXAMPLES

4.1 A frame structure

The detailed process on performing the proposed substructuring method is illustrated by a frame structure in this example. The frame is modeled by 160 two-dimensional beam elements with 2.5m long as in Fig. 1. The material constants are chosen as: bending rigidity \( (EI) = 170 \times 10^6 \text{ Nm}^2 \), axial rigidity \( (EA) = 2500 \times 10^6 \text{ N} \), mass per unit length \( (\rho A) = 110 \text{ kg/m} \), and Poisson's ratio \( = 0.3 \).
The Young’s module of one column element in Substructure 2 (denoted in Fig. 1) is chosen as the design parameter $r_1$, and the eigensensitivity of the first 20 modes of the global structure with respect to this design parameter can be calculated with the following three main steps:

1. The frame is divided into three substructures ($NS=3$) as shown in Fig. 1.

2. Each substructure is analyzed individually. The first 50 modes in each substructure are retained as master modes. The derivative matrices regarding the first 50 modes are calculated as:

   (a) For Substructure 2 (includes the design parameter $r_1$), the derivative matrices of eigensolutions are calculated as: $\frac{\partial \Lambda_m^{(2)}}{\partial r_1}$, $\frac{\partial \Phi_m^{(2)}}{\partial r_1}$, (m=50) with Nelson’s method, and the derivative matrix of residual flexibility is: $\frac{\partial F_c^{(2)}}{\partial r_1}$ according to Eq. (4d);

   (b) For Substructures 1 and 3 (exclude the design parameter $r_1$), the derivative matrices of eigensolutions and residual flexibility are zeros as: $\frac{\partial \Lambda_m^{(j)}}{\partial r_1} = 0$, $\frac{\partial \Phi_m^{(j)}}{\partial r_1} = 0$, $\frac{\partial F_c^{(j)}}{\partial r_1} = 0$, ($j=1,3$).

3. The substructures are reassembled into the global structure. The primitive form of the derivative matrices is constructed as:

   \[
   \frac{\partial \Lambda_m}{\partial r_1} = \text{Diag} \left( \frac{\partial \Lambda_m^{(i)}}{\partial r_1}, \frac{\partial \Lambda_m^{(2)}}{\partial r_1}, \frac{\partial \Lambda_m^{(3)}}{\partial r_1} \right), \quad \frac{\partial \Phi_m}{\partial r_1} = \text{Diag} \left( \frac{\partial \Phi_m^{(i)}}{\partial r_1}, \frac{\partial \Phi_m^{(2)}}{\partial r_1}, \frac{\partial \Phi_m^{(3)}}{\partial r_1} \right), \quad \frac{\partial F_c}{\partial r_1} = \text{Diag} \left( \frac{\partial F_c^{(i)}}{\partial r_1}, \frac{\partial F_c^{(2)}}{\partial r_1}, \frac{\partial F_c^{(3)}}{\partial r_1} \right)
   \]

   According to Eq. (12), the item $\frac{\partial \mathbf{z}_i}{\partial r_1}$ can be computed. Up to now, the eigensensitivity of the global structure with respect to parameter $r_1$ can be formed according to Eq. (4a) and Eq. (6).

Following the three steps, the eigensensitivity of the first 20 modes with respect to the parameter $r_1$ are calculated. For comparison, the eigensensitivity is also computed with the
traditional global method (Nelson, 1976), which are regarded as the accurate value. The results from the proposed substructuring method and the traditional global method are compared in Table 1. In this table, Correlation of Eigenvector Derivatives (COED) indicates the similarity of the eigenvector derivatives between the Nelson’s method and the proposed substructuring method, and given by:

\[
\text{COED} \left( \frac{d\Phi}{dr_i}, \frac{d\Phi}{dr_i} \right) = \frac{\left( \frac{d\Phi}{dr_i} \right)^T \left( \frac{d\Phi}{dr_i} \right)}{\left( \frac{d\Phi}{dr_i} \right) \left( \frac{d\Phi}{dr_i} \right)^T} \tag{13}
\]

in which, \( \frac{d\Phi}{dr_i} \) and \( \frac{d\Phi}{dr_i} \) represents the eigenvector derivatives obtained by Nelson’s method and by the proposed substructuring method respectively.

Table 1. The eigensensitivity with respect to the design parameter \( r_i \)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Eigenvalues</th>
<th>Eigenvector</th>
<th>Relative error</th>
<th>COED</th>
</tr>
</thead>
<tbody>
<tr>
<td>Global method</td>
<td>Substructuring method</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.8756</td>
<td>0.8756</td>
<td>0.00%</td>
<td>0.9958</td>
</tr>
<tr>
<td>2</td>
<td>3.6014</td>
<td>3.6014</td>
<td>0.00%</td>
<td>0.9984</td>
</tr>
<tr>
<td>3</td>
<td>3.2932</td>
<td>3.2939</td>
<td>0.02%</td>
<td>0.9964</td>
</tr>
<tr>
<td>4</td>
<td>47.1077</td>
<td>47.1103</td>
<td>0.01%</td>
<td>0.9967</td>
</tr>
<tr>
<td>5</td>
<td>67.9608</td>
<td>68.0655</td>
<td>0.15%</td>
<td>0.9988</td>
</tr>
<tr>
<td>6</td>
<td>274.9365</td>
<td>275.6273</td>
<td>0.25%</td>
<td>0.9984</td>
</tr>
<tr>
<td>7</td>
<td>205.7972</td>
<td>205.8071</td>
<td>0.00%</td>
<td>0.9973</td>
</tr>
<tr>
<td>8</td>
<td>715.4812</td>
<td>720.8485</td>
<td>0.75%</td>
<td>0.9972</td>
</tr>
<tr>
<td>9</td>
<td>633.383</td>
<td>637.1267</td>
<td>0.59%</td>
<td>0.9979</td>
</tr>
<tr>
<td>10</td>
<td>523.1332</td>
<td>526.1615</td>
<td>0.58%</td>
<td>0.9983</td>
</tr>
</tbody>
</table>

Table 1 demonstrated that, the errors of most eigenvalue derivatives are less than 1%, and the similarity of eigenvector derivatives is above 0.987. It means that, when the first 50 eigensolutions in each substructure are chosen as master, the proposed substructuring method can reach a good precision in calculating the eigensensitivity for the first 20 modes. The eigensensitivity of the global structure is formed on the particular substructure and a reduced eigenequation. The computation efficiency might be improved, and will be investigated in the following example.

4.2 A practical bridge

To further investigate the property of the proposed substructuring method, such as efficiency and division formation, a practical bridge over the Balla Balla River in Western Australia is discussed here. The FE model of this bridge (Fig. 2) has 907 elements, 947 nodes, and 5420 DOFs in total (Xia et al, 2008).

The designed parameters include the Young’s modulus of 10 elements, which are randomly chosen from the 907 elements. The computation time on calculating the eigensensitivity of the first 20 modes with respect to the ten elemental parameters is denoted as an indicator of efficiency. Certainly, the division formation of the substructures affects the computation efficiency. To investigate the effect of division formation, the bridge is averagely divided into 5
substructures, 8 substructures, 11 substructures and 15 substructures along the longitudinal direction.

![Figure 2. The FE model of the Balla Balla River Bridge.](image)

Selecting different master modes in each substructure will influence the accuracy of the proposed substructuring method, and undoubtedly consuming different computation time. If the errors of eigenvalue derivatives are required below 3%, there are 80 master modes retained in each substructure with the division formation of 5 substructures, 60 master modes with 8 substructures, 50 master modes with 11 substructures, and 50 master modes with 15 substructures. The total computation time in calculating the eigensensitivity of the first 20 modes of the global structure with the four division schemes is compared in Fig. 3.

![Figure 3. The computation time with four division formations.](image)

From Fig. 3, it can be found that:

1. Since the eigensensitivity can be formed based on only particular substructure and the reduced eigenequation, the proposed method can improve the computation efficiency. For example, with the traditional global method, one has to obtain the eigensensitivity of the global structure from a large system matrix which has the size of $5420 \times 5420$. However, with the proposed substructuring method, when the global structure is divided into 11 substructures, the eigensensitivity can be instead by analyzing a substructure with about 500 DOFs and a reduced eigenequation with the size of $550 \times 550$.

2. The computation efficiency heavily depends on the substructure division and the master modes selection. More master modes undoubtedly contribute to higher precision, but consume more computation time. The master modes depend on the precision requirement. The influence of division formation is more complicated. For example, dividing the global
structure into 5 substructures or 8 substructures takes longer computation time than that of 11 substructures. This is because handling large substructures takes up much computation resource. However, the smaller substructures are not always favorable. The division formation of 15 substructures is not as efficient as that of 11 substructures, since much excessive substructures lead to large connection matrix $C$ and the primitive matrices of the substructures. In that case, the assembly of the ‘substructures’ to the ‘global’ structure will take longer time. One should trade off the number of substructures and the size of each substructure. Nevertheless, trial and error approach can be used to determine the optimal substructure formation before application to model updating process.

5 CONCLUSION

In this paper, a substructuring scheme is utilized to calculate the first-order derivatives of eigensolutions. A frame structure and a practical bridge are analyzed to verify that the proposed method can achieve a good accuracy when proper master modes are retained.

When an elemental parameter is changed within a substructure, only that particular substructure needs to be reanalyzed, and other substructures are untouched. The eigensensitivity of the global structure is assembled by constraining on the substructures. Therefore, the computation efficiency is improved. Additionally, the division formation heavily influences the computation efficiency, which should be considered cautiously in advance and merits further investigations.

ACKNOWLEDGMENTS

The work described in this paper is supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. PolyU 5321/08E).

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